The forces of inertial oscillations

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By starting with a free particle and successively adding constraints, it is shown that the free motion of a particle constrained to the earth's surface is inertial, despite statements in the literature, and that an observer on this particle would not measure a force tangent to the Earth's surface. However, if the observer extended his measurements to include the direction normal to the Earth's surface, he would detect an oscillating force.

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1. Introduction

The primitive equations in oceanography and atmospheric science describe a rich set of physical phenomena, but their complexity makes analytical solutions exceptionally rare. Only by simplifying the equations do we hope to find such solutions and fully explain the forcing involved in the resulting solution. The inertial oscillation problem exemplifies this approach by reducing the primitive equations to describe the motion of a particle confined to the surface of the earth. While inertial motions are a ubiquitous facet of ocean currents (Gill 1982; Pollard and Millard 1970), the interest in this problem lies also in its simplification of the underlying forcing. Indeed, when mathematician and meteorologist Francis John Welsh Whipple (Whipple 1917) stated the problem in 1917 he wrote "...the author considers that a knowledge of the motion of such a particle will prove a useful preliminary to a proper understanding of the more complicated motion which actually occurs in winds, where the air particles have other forces besides that of gravity acting upon them." Nearly a century later the exact solutions for the particle constrained to the earth have been found (Pennell and Seitter 1990; Paldor and Sigalov 2001) but do not appear to be well known and the forces responsible for the motion have continued to be debated and discussed (Stommel and Moore 1989; Durran 1993; Ripa 1997; Persson 1998; Phillips 2000).

This manuscript corrects a fundamental and widespread misconception concerning inertial oscillations and whether or not inertial oscillations are accurately described by the term 'inertial'. It's important to note that this is not merely an issue of semantics. The name 'inertial oscillation' suggests that the observed oscillatory motion of the particle is explained entirely by inertial forces, apparent only because of our choice of reference frame. We can reframe this issue as a simple thought experiment by asking whether or not an observer inside a small laboratory set in a perfect 'inertial oscillation' could design an experiment to distinguish his laboratory from an inertial frame. In a more practical sense, we are asking whether an accelerometer set in perfect inertial motion measure accelerations? If the accelerometer does not measure an acceleration then the name 'inertial oscillation' is appropriate, but by contrast if an acceleration is detected, then the name is fundamentally wrong and should be changed.

There is a seductively simple, but incorrect, proof which appears to show that there is in fact a force in the inertial frame. The f-plane approximation to the horizontal equations of motion are simply

$$\frac{d\mathbf{v_r}}{dt} + 2\omega \times \mathbf{v_r} = 0 \tag{1}$$

where $\mathbf{v_r}$ is the two-dimensional velocity vector of the according to the observer on the earth and ω is the angular frequency of rotation of the earth. By simple kinematic computation the relationship between acceleration in the rotating and fixed frame is given by

$$\frac{d\mathbf{v_f}}{dt} = \frac{d\mathbf{v_r}}{dt} + 2\omega \times \mathbf{v_r} + \omega \times (\omega \times \mathbf{r}). \tag{2}$$

If we take equation (2) and apply equation (1) we see that in the fixed frame,

$$\frac{d\mathbf{v_f}}{dt} = \omega \times (\omega \times \mathbf{r}). \tag{3}$$

Because $\mathbf{v_f}$ is the velocity in the fixed frame, then by Newton's second law equation (3) shows there to be a force acting on the particle. Although the force in equation (3) appears to be centrifugal, Durran (1993) correctly points out that the term is actually a component of gravity. From this we might conclude that because there is a force in the fixed frame, the particle's motion is not inertial. This result suggests a profound difference from the original claim that an accelerometer would not detect any accelerations. By this reasoning it is now repeated in the literature and the community that these oscillations are not inertial, in contradiction with the claim that an accelerometer would not detect any accelerations.

Starting with free particle and successively adding constraints we will show that the particle's motion is in fact inertial and the accelerometer trapped in inertial motion would not measure an acceleration. The 'force' identified in equation (3) is a component of gravity, fundamentally indistinguishable from other inertial forces that result from our choice of reference frame. General relativity tells us that an astronaut in orbit around the earth can measure no accelerations, despite the curvature of his orbit and the presence of gravity. For the same reason a particle in an inertial oscillation on the surface of the earth will measure an acceleration only in the vertical direction, but not in the direction of its motion, due to changes in the strength of the normal force throughout the course of an oscillation. Inertial oscillations are truly inertial in the local horizontal and therefore deserving of the name. On the other hand, it will also be shown that the name inertial oscillation suffers from ambiguity and that a more descriptive name can be devised based on the problem constraints.

To understand the forcing we will use the Lagrangian (Lagrange function) to help write down and understand the forces involved. The advantage to this technique is its simplicity: small changes in our assumptions about the system effortlessly show changes in the forcing. Starting with the description of a free particle in spherical coordinates and adding a new assumption at each step, we examine two other systems nearly identical to the particle on a earth: a particle in orbit and a particle constrained to a sphere. The particle in orbit introduces gravity into the problem without the complication of constraints and then by constraining the particle to the sphere, the effects of the constraint force are more easily isolated. By considering these two other systems first, we will finally highlight the subtle differences that arise in physical systems on the earth and have a clear understanding of source of each force. For completeness, the exact solutions – extending the work of Pennell and Seitter (1990) and Paldor and Sigalov (2001) – are presented in the Appendix.

2. Free Particle, Inertial Frame

The Lagrangian for a conservative mechanical system is the kinetic energy minus the potential energy, $\mathcal{L} = KE - PE$. We find the path that minimizes the action, $S = \int \mathcal{L}\, dt$, using the Euler-Lagrange equations, $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}$, and recover the equations of motion exactly as if we'd written out the forces explicitly using Newton's second law.

We use the coordinate system (ϕ_I, θ, r) in the inertial frame where $\hat{\phi_I}$ points eastward in the direction of increasing longitude and $\hat{\theta}$ points northward, in the direction of increasing latitude. To an observer standing on the Earth, these coordinates locally look like (x, y, z): east, north and up. The Lagrangian for a free particle is the kinetic energy of the particle,

$$\mathcal{L}^{\mathrm{I}}_{\mathrm{free}} = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + \frac{1}{2}r^2\dot{\phi_I}^2\cos^2\theta \tag{4}$$

where dot is used to indicate the time derivative, e.g., $\dot{r} = \frac{dr}{dt}$. By applying the Euler-Lagrange equations to (4), the equations of motion are found to be

$$\begin{bmatrix} a_{\phi_I} \\ a_\theta \\ a_r \end{bmatrix} \equiv \begin{bmatrix} r\ddot{\phi}_I\cos\theta + 2\dot{r}\dot{\phi}_I\cos\theta - 2r\dot{\phi}_I\dot{\theta} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} + r\dot{\phi}_I^2\sin\theta\cos\theta \\ \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}_I^2\cos^2\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{5}$$

which we've also used to define the acceleration vector $\mathbf{a}$, in terms of the coordinates (ϕ_I, θ, r) and their time derivatives. Equation (5) is written to resemble Newton's second law $\mathbf{a} = \mathbf{f}$ where $\mathbf{a}$ is the acceleration and $\mathbf{f}$ is the specific force (zero, in this case). Although these are the equations of motion for a free particle, they appear rather complicated because they're written in a coordinate system not well suited for the problem, spherical coordinates. It is important to keep in mind that all of these terms are necessary simply to describe an unforced, and therefore unaccelerated, particle moving in a straight line. Only additional constraints added to the Lagrangian will result in terms that appear as forces.

3. Free Particle, Rotating Frame

In order to identify the so-called "inertial forces", we need to transform equation (4) into the rotating frame. We apply the transformation $\phi_I \mapsto \phi + \omega t$ where ω is the angular rotation frequency of the earth and ϕ is the new longitudinal coordinate. The Lagrangian becomes,

$$\mathcal{L}^{\mathrm{R}}_{\mathrm{free}} = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + \frac{1}{2}r^2\left(\dot{\phi}^2 + 2\omega\dot{\phi} + \omega^2\right)\cos^2\theta, \tag{6}$$

from which we obtain the following equations of motion,

$$\begin{bmatrix} a_\phi \\ a_\theta \\ a_r \end{bmatrix} = \begin{bmatrix} -2\dot{r}\omega\cos\theta + 2r\dot{\theta}\omega\sin\theta \\ -2r\dot{\phi}\omega\sin\theta\cos\theta - r\omega^2\sin\theta\cos\theta \\ 2r\dot{\phi}\omega\cos^2\theta + r\omega^2\cos^2\theta \end{bmatrix}. \tag{7}$$

To the rotating observer believing he is in an inertial frame, everything on the left-hand-side of the equations appear as appropriate acceleration terms for his chosen spherical coordinate system, while everything on the right-hand-side appear to be forces. These are the so-called "inertial forces". The four terms on the right containing a velocity are the Coriolis force, while the two terms only dependent on position are the centrifugal force.

Imagine that vector $\mathbf{r_I}$ describes the motion of a spaceship far between the stars, which is therefore experiencing no forcing, and satisfies equation (5). The rotating observer, in contrast, describes the motion of the spaceship with vector $\mathbf{r_R}$ satisfying equations (7). In a practical sense what makes these forces in equation (7) inertial is that when a person on the spaceship measures his acceleration with an accelerometer, he will find his acceleration to be zero. By Newton's second law the person in the spaceship would correctly conclude no forces are acting on him, despite the rotating observer's belief to the contrary.

4. Central Gravitational Field, Rotating Frame

Let's extend the free particle Lagrangian of equation (6) by including a central gravity field similar to the earth's,

$$\begin{aligned}\mathcal{L}^{\mathrm{R}}_{\text{gravity}} =& \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 \\ &+ \frac{1}{2}r^2\left(\dot{\phi}^2 + 2\omega\dot{\phi} + \omega^2\right)\cos^2\theta + \frac{GM}{r}.\end{aligned} \tag{8}$$

The equations of motion for this system differ from equation (7) only by the addition of a gravitational force included in the radial acceleration, a_r,

$$\begin{bmatrix} a_\phi \\ a_\theta \\ a_r \end{bmatrix} = \begin{bmatrix} -2\dot{r}\omega\cos\theta + 2r\dot{\theta}\omega\sin\theta \\ -2r\dot{\phi}\omega\sin\theta\cos\theta - r\omega^2\sin\theta\cos\theta \\ 2r\dot{\phi}\omega\cos^2\theta + r\omega^2\cos^2\theta - \boxed{\frac{GM}{r^2}} \end{bmatrix}. \tag{9}$$

The solutions to (9) are also well known and include motions like the International Space Station's (ISS) orbit path and geosynchronous satellites, figure (1).

This system too only involves "inertial forces." From Einstein's equivalence principle we know that we could not conduct an experiment to distinguish between an elevator at rest on the earth and an elevator accelerating by rocket at 9.8 m s^{-2}. So while Newton would have argued that we introduced a force of gravity into equation (9), by the equivalence principle we know that this is really just another "inertial" term. As a practical consequence, astronauts aboard the ISS do not detect acceleration with their accelerometers. Because they are in free fall, they are in an inertial reference frame, locally indistinguishable from the reference frame of the inertial observer of section (3) *without* a central gravitational field. This reflects an evolution in our understanding of forces from general relativity because we cannot design a local experiment to distinguish between the cases with and without this term. Gravity is now more appropriately considered a description of the local geometry of the problem, and not a force. This is an important point at the heart of general relativity, see chapter 4.3 of Wald (1984) or chapter 1 of Misner *et al.* (1973).

By this reasoning then, the oscillations of satellites around the earth or oscillations of satellites in near geosynchronous orbit could justly be named "inertial oscillations."

5. Particle on a Rotating Sphere

We can constrain the particle to the surface of a sphere simply by setting $r = R$. However, a more fruitful technique is to apply our constraint with a Lagrange multiplier, denoted $\lambda^{\mathrm{R}}_{\text{sphere}}$, so that we analyze the resulting force required to keep the particle on the surface of the sphere. $\lambda^{\mathrm{R}}_{\text{sphere}}$ is treated as variable just like ϕ, θ and r, but it multiplies the constraint, namely that $r - R = 0$. With this addition, the Lagrangian becomes

$$\begin{aligned}\mathcal{L}^{\mathrm{R}}_{\text{sphere}} =& \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + \frac{1}{2}r^2\left(\dot{\phi}^2 + 2\omega\dot{\phi} + \omega^2\right)\cos^2\theta \\ &+ \frac{GM}{r} + \lambda^{\mathrm{R}}_{\text{sphere}}(r - R).\end{aligned} \tag{10}$$

In the inertial frame equation (10) would describe geodesic motion on the surface of a sphere, the path of the great circles. This system was analyzed by McIntyre (2000) in

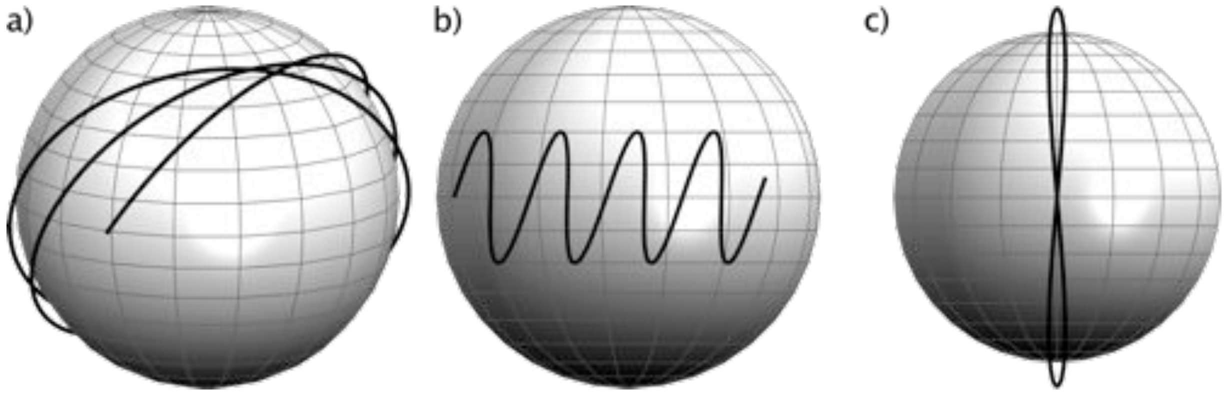

Figure 1. Three example particle paths in a central gravitational field as viewed from a rotating reference frame. Panel a) shows an orbit that approximates the International Space Station, b) is a small deviation from geosynchronous orbit and c) is a geosynchronous orbit with an inclination. All three are inertial and exhibit oscillatory motion.

both the rotating and the inertial frame. The equations of motion now include a fourth equation using $\lambda^{R}_{\text{sphere}}$ as a variable in the Euler-Lagrange equations,

$$\begin{bmatrix} a_\phi \\ a_\theta \\ a_r \\ r \end{bmatrix} = \begin{bmatrix} -2\dot{r}\omega\cos\theta + 2r\dot{\theta}\omega\sin\theta \\ -2r\dot{\phi}\omega\sin\theta\cos\theta - r\omega^2\sin\theta\cos\theta \\ 2r\dot{\phi}\omega\cos^2\theta + r\omega^2\cos^2\theta - \frac{GM}{r^2} + \lambda^{R}_{\text{sphere}} \\ R \end{bmatrix}. \tag{11}$$

By applying the constraint condition $r = R$ and using the definitions in equation (5), we can now use the a_r component of equation (11) to find the force of constraint,

$$\lambda^{R}_{\text{sphere}} = \overbrace{\frac{GM}{R^2}}^{\text{gravity}} - \overbrace{2R\dot{\phi}\omega\cos^2\theta}^{\text{coriolis}} - \overbrace{R\omega^2\cos^2\theta}^{\text{centrifugal}} - \underbrace{R\dot{\theta}^2 - R\dot{\phi}^2\cos^2\theta}_{\text{geometric}}. \tag{12}$$

The force of constraint in equation (12) is synonymous with the term normal force (represented $\mathbf{F_N}$) because it is necessarily perpendicular to the resulting motion. The normal force is a real force and stands in as a proxy for the collection of electromagnetic forces that keep our particle from plunging into the depths of this spherical earth. The accelerations caused by this force are detectable by our accelerometer. The first term is exactly opposite and equal to the force of gravity, completely negating its effect. The next two terms in equation (12) oppose components of the inertial forces, the Coriolis and centrifugal force. This real force that opposes the inertial centrifugal force is called the centripetal force. The last two terms are perhaps unexpected. They represent the constraint force applied against the particle's inertia preventing it from moving in a straight, unaccelerated path. These two terms would be present even in the inertial frame ($\omega = 0$) and without gravity ($G = 0$); they distinguish geodesic motion in Euclidean space from geodesic motion on a sphere and could be called geometric forces.

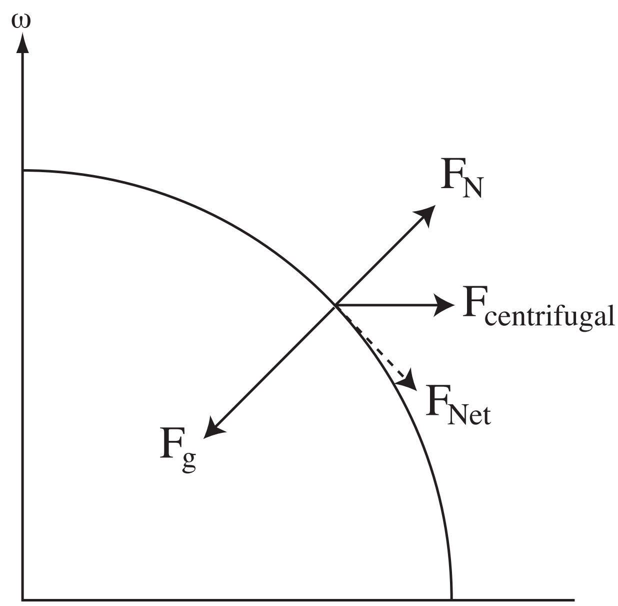

Figure 2. Force diagram for a particle initially at rest on a rotating sphere. The net force points towards the equator, but only the normal force, $\mathbf{F_N}$, is a real, measurable force.

Consider the forces acting on the particle initially at rest by setting $(\dot{\phi}, \dot{\theta}, \dot{r}) = (0, 0, 0)$ in equation (11). In this case the only forces acting on the particle are gravity, the normal force, and the centrifugal force, see figure (2). However, we can see from equation (11) that a component of the centrifugal force still points in the $-\hat{\theta}$ direction, pushing the particle towards the equator. This net force is exactly why we know that this Lagrangian does not correctly describe a particle on the earth's surface (objects initially at rest on the earth's surface do not roll equatorward).

The resulting motion shown in figure (3) is again periodic, so could this too be described as inertial oscillations? We have established that there exists a force of constraint, but how would the observer moving with the particle experience this motion? Assume the observer starts initially at rest with respect to the rotating surface. We can see from figure (3) that the observer will oscillate about the equator, drifting slowly westward. Because there are only inertial forces in the $\hat{\phi}$ and $\hat{\theta}$ directions, the observer will not detect any accelerations in those directions. However, the observer's local vertical acceleration will show change, directly computable with equation (12). There will be a constant acceleration detected opposite and equal to gravity, but in addition there is a position dependent centripetal force as well as the other velocity dependent geometric forces in equation (12). To the three-dimensional observer, any motion described by this system has detectable forces and therefore is not inertial motion. However, the spirit of the problem is to restrict the motion, and perhaps therefore also the observations, to only two dimensions. In the remaining two dimensions the motion only contains inertial forces and it seems reasonable to describe the oscillatory motion as inertial oscillations.

6. Particle on Earth

In this final case we want to consider a particle under the influence of gravity and constrained to the surface of the earth. The gravitational potential is parameterized as the more general $G(r,\theta)$ instead of the potential for a point mass (and sphere), $-\frac{GM}{r}$. The surface of the earth is described by some function $f(r,\theta)$ and therefore our constraint is described by $f(r,\theta)=k$. The Lagrangian in the rotating frame is

$$\begin{aligned}\mathcal{L}^{\mathrm{R}}_{\mathrm{earth}} =& \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + \frac{1}{2}r^2\left(\dot{\phi}^2 + 2\omega\dot{\phi} + \omega^2\right)\cos^2\theta \\ &- G(r,\theta) + \lambda^{\mathrm{R}}_{\mathrm{earth}}\left(f(r,\theta) - k\right).\end{aligned} \qquad (13)$$

This looks identical to equation (10) except for the generalizations of the gravitational potential and the constraint force. The key difference, shown in Ripa (1997), is that

$$V_{\mathrm{geo}}(r,\theta) \equiv G(r,\theta) - \frac{1}{2}r^2\omega^2\cos^2\theta \qquad (14)$$

must be equal to a constant at the surface of the earth. According to NIMA (2000) V_{geo} is an effective potential called the *total gravity potential* and a surface of constant potential is a *geopotential* surface, or *geop*. In NIMA (2000) it also stated that "the geoid is that particular geop that is closely associated with the mean ocean surface." So this means that V_{geo} describes the surface of the earth, which is exactly what we said $f(r,\theta)$ does! This should simplify things.

The idea that the earth's surface is defined by the geopotential can be understood by considering the previous example of the particle on the rotating sphere. If the earth were a sphere as described by the system (10), then all particles on its surface would roll towards the equator. So if we imagine the earth were entirely fluid and in static equilibrium everything would have already rolled towards the equator; its shape would have relaxed to this minimal potential energy state. The reality is that over the oceans it's pretty close.

However, now we have a problem with our coordinates. A good set of coordinates for this problem would be defined with one basis vector normal to the surface and the other two basis vectors tangent to the surface. Our spherical coordinates do this for a sphere, but not for a geopotential surface which is clearly a function of both θ and r. The vector $\hat{\phi}$ points tangent to the surface, but $\hat{\theta}$ and $\hat{r}$ have components both tangent and normal to the surface. If we

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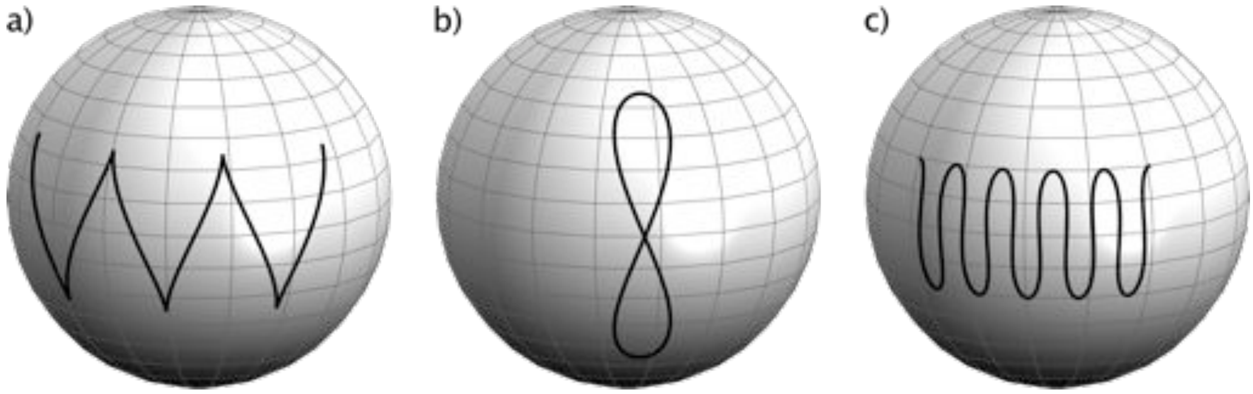

Figure 3. Three example particle paths on a sphere as viewed from a rotating reference frame of an eastward rotate sphere. Panel a) depicts the path of a particle released from rest in the rotating reference frame, which subsequently drifts to the west. Panels b) and c) show different choices of initially eastward velocities. All three are inertial tangent to the surface, and oscillatory.

approximate our geopotential as an ellipse, and then use elliptic coordinates, we can achieve the desired results. This turns out not to change the form of the Lagrangian significantly. The key observation is that if we use a good set of coordinates where say, $\hat{\xi}$ is normal to the geopotential, then V_{geo} is entirely a function of ξ. The change in direction of the forcing from the difference between the geometry of a spheroid and an ellipsoid is negligible (Gill 1982). At this point we will then redefine our coordinates so that $\hat{r}$ is normal to the geopotential and $\hat{\theta}$ is perpendicular to both $\hat{r}$ and $\hat{\phi}$. This also allows us to restate our constraint as constant r, rather than constant V_{geo}. The rotating observer standing on the earth experiences $-\frac{\partial V_{\text{geo}}}{\partial r}$ as the total force of gravity, so we use equation (14) and neglecting those small geometric deviations to rewrite equation (13) as

$$\begin{aligned}\mathcal{L}^{\text{R}}_{\text{earth}} =& \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + \frac{1}{2}r^2\left(\dot{\phi}^2 + 2\omega\dot{\phi}\right)\cos^2\theta \\ &- V_{\text{geo}}(r) + \lambda^{\text{R}}_{\text{earth}}\left(r - R\right),\end{aligned} \quad (15)$$

where R is the approximate radius of the earth.

At this point if we were to ignore the force of constraint and simply proceed to set $r = R$ we would recover the same Lagrangian as in Ripa (1997), and therefore produce the same force diagrams drawn in figure 1 of both Ripa (1997) and Durran (1993). However, let's proceed with the force of constraint still under consideration and apply the Euler-Lagrange equations to examine the resulting force. In the rotating frame we find

$$\begin{bmatrix} a_\phi \\ a_\theta \\ a_r \\ r \end{bmatrix} = \begin{bmatrix} -2\dot{r}\omega\cos\theta + 2r\dot{\theta}\omega\sin\theta \\ -2r\dot{\phi}\omega\sin\theta\cos\theta \\ 2r\dot{\phi}\omega\cos^2\theta - \frac{\partial V_{\text{geo}}}{\partial r} + \lambda^{\text{R}}_{\text{earth}} \\ R \end{bmatrix}. \quad (16)$$

Equation (16) looks almost identical to the equations of motion on a sphere, equation (11). The centrifugal and gravitational force in the radial component of equation (11) are combined in the definition of V_{geo}. However, the centrifugal force in the $\hat{\theta}$ component of equation (11) is missing from equation (16). Qualitatively at least this looks a lot more like the surface of the earth as we know it: objects initially at rest do not roll towards the equator. The constraint force is

$$\lambda^{\text{R}}_{\text{earth}} = \left.\frac{\partial V_{\text{geo}}}{\partial r}\right|_{r=R} - 2R\dot{\phi}\omega\cos^2\theta - R\dot{\theta}^2 - R\dot{\phi}^2\cos^2\theta, \quad (17)$$

which appears nearly identical to the constraint force on the sphere, equation (12). For the particle initially at rest, the constraint force evaluates to

$$\begin{aligned}\lambda^{\text{R}}_{\text{earth}} &= \left.\frac{\partial V_{\text{geo}}}{\partial r}\right|_{r=R} \\ &= \left.\frac{\partial G}{\partial r}\right|_{r=R} - R\omega^2\cos^2\theta\end{aligned} \quad (18)$$

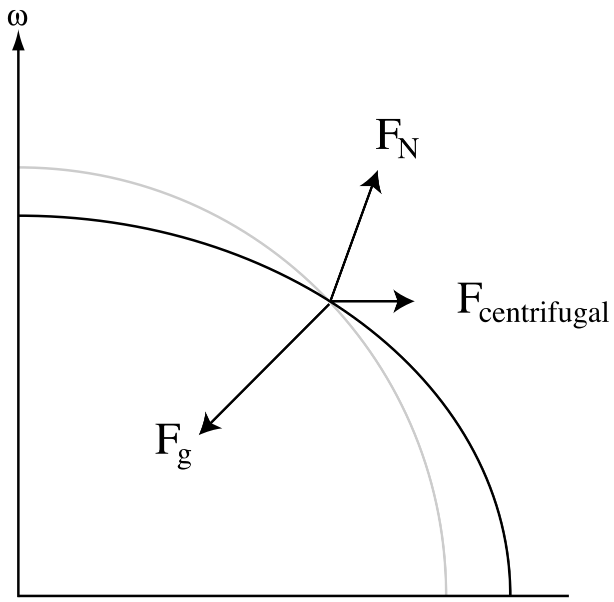

Figure 4. Force diagram for a particle initially at rest on the earth. All forces balance, so the particle will remain at rest in this reference frame. The outline of a sphere is shown in gray.

and is depicted in figure (4). Because the constraint force is the only real force in this system, the motion is inertial in the $\hat{\phi}$ and $\hat{\theta}$ directions, just as with the particle on the sphere. So where's the discrepancy with the derivation in section 1?

Consider the Lagrangian in the inertial frame, but without using equation (14) to replace the gravitational potential with the total gravity potential,

$$\begin{aligned} \mathcal{L}^{\mathrm{I}}_{\mathrm{earth}} =& \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + \frac{1}{2}r^2\dot{\phi_I}^2\cos^2\theta \\ & - G(r,\theta) + \lambda^{\mathrm{I}}_{\mathrm{earth}}\left(r - R\right). \end{aligned} \tag{19}$$

Applying the Euler-Lagrange equations we find that

$$\begin{bmatrix} a_{\phi_I} \\ a_\theta \\ a_r \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{r}\frac{\partial G}{\partial \theta} \\ -\frac{\partial G}{\partial r} + \lambda^{\mathrm{I}}_{\mathrm{earth}} \\ R \end{bmatrix}. \tag{20}$$

Equation (20) shows that unlike the case of the sphere, gravity has a component tangent to the surface and in fact, from equation (14), we know the magnitude of this component is exactly $r\omega^2 \sin\theta\cos\theta$. The gravitational forces of equation (20) are however, not real forces and it is only the constraint force $\lambda^{\mathrm{I}}_{\mathrm{earth}}$ found in the $\hat{r}$ direction that is measured by the accelerometer. Solving for $\lambda^{\mathrm{I}}_{\mathrm{earth}}$ in equation (20),

$$\lambda^{\mathrm{I}}_{\mathrm{earth}} = \frac{\partial G}{\partial r} - R\dot{\theta}^2 - R\dot{\phi_I}^2\cos^2\theta, \tag{21}$$

we see that the constraint force balances only the radial component of gravity, $\hat{r}$, and not its horizontal components. A particle at rest in the rotating frame, $(\dot{\phi}_I, \dot{\theta}, \dot{r}) = (\omega, 0, 0)$, has a normal force of

$$\lambda^{\mathrm{R}}_{\mathrm{earth}} = \left.\frac{\partial G}{\partial r}\right|_{r=R} - R\omega^2\cos^2\theta. \tag{22}$$

Equations (18) and (22) are identical, meaning the rotating observer and inertial observer predict the same reading on the accelerometer. The forces as they appear from the inertial frame are drawn in figure (4). For the inertial observer having a non-zero net force looks correct. The particle is rotating around the polar axis at speed ω and therefore is accelerating inward.

Following the incorrect logic of the proof in section 1, consider what would happen if we believed that the $\hat{\theta}$ component of gravity in equation (20) really was a force. This would suggest that accelerometer measurement by an observer standing on the earth would point not in the local vertical, but would point slightly equatorward! This is certainly not our experience and so we must reject the notion that gravity is a true force.

The accelerometer trapped in inertial oscillations would therefore not detect any motion in the $\hat{\phi}$ or $\hat{\theta}$ directions. However, like the particle on the sphere, the accelerometer would detect changes in acceleration in the local vertical due to the changing magnitude of the normal force. So again, we could justly call these oscillations are inertial if we restrict measurements to the two dimensions of the earth's surface. Representative solutions are shown in figure (5)

Notice that the constraint force (17) provides enough information that an observer in inertial motion measuring acceleration in the vertical would be able to deduce his motion. The first term in the constraint force is constant, opposite and equal to the total gravity, and the last two terms are proportional to the energy, and are therefore also constant. The frequency and magnitude of the oscillations of the second term would allow the observer to determine the latitude and speed of his motion at any instant, given the solutions shown in the appendix.

7. Conclusions

The name "inertial oscillations" turns out not to be so bad after all. The "force" shown to exist in the $\hat{\theta}$ direction is gravity and by Einstein's equivalence principle it is therefore not measurable by an accelerometer. This additional term should not strictly be considered a force, but in fact a description of the local geometry. In our daily lives we often mistakenly think of gravity as the force we feel against our feet, but the reality is that that it's the earth pushing us up. The result of this is that an accelerometer trapped in an inertial oscillation would not detect any accelerations in the direction tangent to earth's surface and only measurement in the local vertical would indicate that the particle is being forced. So given the lack of observable forces the oscillations described by the Lagrangian in equation (15) are inertial.

Is inertial oscillation really an appropriate name for this phenomenon? In all four systems we examined, the motion could be described as inertial and all four systems exhibit oscillations, so the name "inertial oscillation" isn't particularly descriptive. Durran (1993) suggested "constant angular momentum oscillation", but this too suffers from ambiguity. Because ϕ does not explicitly enter into any of the four Lagrangians we consider, this means that angular momentum is conserved in all four cases. The key feature distinguishing oscillations on the earth from the other three systems is the constraint of the particle to the geopotential. In order to satisfy equation (14) the direction of the total gravity force and the gravitational force always differ by a centrifugal component, resulting in an apparent net force in the inertial frame. This will always be a feature of a geopotential of the Earth or any other rotating planet and therefore the name "geoinertial oscillation" is perhaps more descriptive.

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A. Exact Solutions

Numerical solutions to the inertial oscillation problem have been previously considered (Paldor and Killworth 1988), but the exact solutions were first found by directly integrating the equations of motion (Pennell and Seitter 1990) and then later by use of action-angle variables (Paldor and Sigalov 2001). The exact solutions can also be found by directly integrating the conserved quantities implied by the Lagrangian in equation (15), which is the approach taken here. From the three different solutions for the path $(\phi(t), \theta(t))$, we can also compute, for the first time, the particle velocity, $(u(t), v(t))$ for the different solutions in closed form. A Matlab script to compute the exact solutions for particle path, velocity, and frequency of oscillation for all three cases is available at http://jeffreyearly.com.

Starting with equation (15), but applying the constraint that $r = R$, the Lagrangian simplifies to

$$\mathcal{L}^{\mathrm{R}}_{\mathrm{earth}} = \frac{1}{2} R^2 \dot{\theta}^2 + \frac{1}{2} R^2 \left(\dot{\phi}^2 + 2\omega\dot{\phi} \right) \cos^2\theta. \qquad (23)$$

Equation (23) has two conserved quantities, energy $E \equiv \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L}$, and angular momentum $L \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$. The energy can be written as

$$E = \frac{1}{2} R^2 \dot{\theta}^2 + \frac{1}{2} R^2 \cos^2\theta \dot{\phi}^2,$$

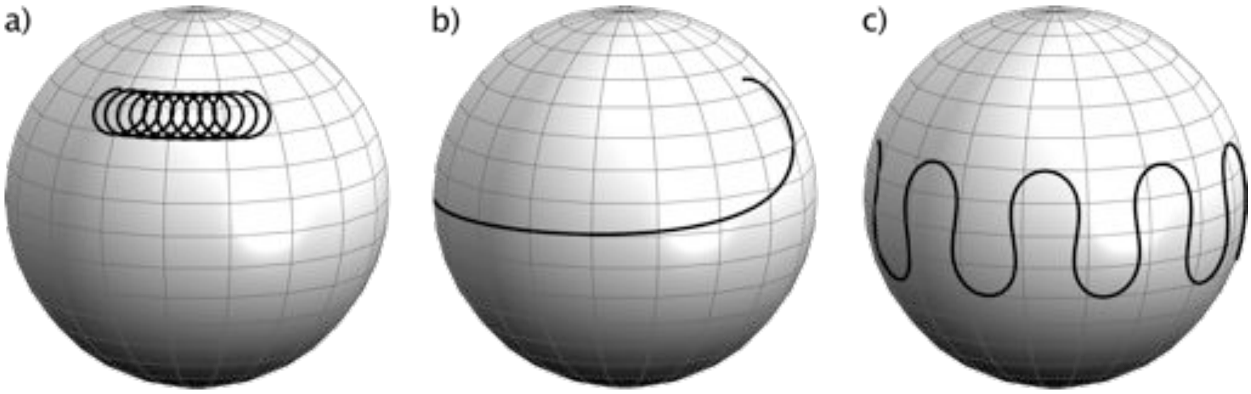

Figure 5. Three example particle paths on the earth as viewed from a rotating reference frame. Just as with the sphere, all three are inertial tangent to the surface and exhibit oscillatory motion.

or in the more familiar $E = \left(u^2 + v^2\right)/2$ if we define velocities $(u, v) = \left(R\dot{\phi}\cos\theta, R\dot{\theta}\right)$. The angular momentum of the particle

$$L = R^2 \cos^2\theta(\dot{\phi} + \omega) \tag{24}$$

can be used to eliminate $\dot{\phi}$ from the energy equation,

$$E = \frac{1}{2}\frac{L^2}{R^2\cos^2\theta} + \frac{1}{2}R^2\dot{\theta}^2 + \frac{1}{2}R^2\omega^2\cos^2\theta - \omega L. \tag{25}$$

Equation (25) is a first order ordinary differential equation that can be used to compute $\theta(t)$ by direct integration. After finding $\theta(t)$, $\phi(t)$ can be found by integrating equation (24).

Solving for $\dot{\theta}$ in equation (25) and then integrating,

$$\omega t = \int_{\theta_0}^{\theta} \frac{\cos\theta}{\sqrt{\frac{2(E+\omega L)}{\omega^2 R^2}\cos^2\theta - \frac{L^2}{\omega^2 R^4} - \cos^4\theta}}\, d\theta. \tag{26}$$

Choosing initial conditions such that the particle is launched with an initially eastward trajectory, $(u_0, v_0) = (R\dot{\phi}_0\cos\theta_0, 0)$, allows us to factor the quartic polynomial under the radical. Writing the conserved quantities E and L in equation (26) in terms of these initial conditions, the integral becomes

$$\omega t = \int_{\theta_0}^{\theta} \frac{\cos\theta}{\sqrt{(\cos^2\theta - \cos^2\theta_0)\left((1+u_\phi)^2\cos^2\theta_0 - \cos^2\theta\right)}}\, d\theta \tag{27}$$

where we've defined $u_\phi = \dot{\phi}_0/\omega$, the ratio of the initial velocity of the particle to the tangential velocity of the earth.

The substitution $s = \sin\theta/\sin\theta_0$ reduces equation (27) to a standard form for inverse elliptic integrals,

$$\sin\theta_0\omega t = \int_1^{\frac{\sin\theta}{\sin\theta_0}} \frac{ds}{\sqrt{(1-s^2)\left(s^2 - k'^2\right)}} \tag{28}$$

where $k'^2 = (1 - (1+u_\phi)^2\cos^2\theta_0)/\sin^2\theta_0$. The parameter k' is called the *complementary modulus* and is related to the *modulus* by $k^2 + k'^2 = 1$ (Byrd *et al.* 1971). The modulus is therefore $k^2 = \cot^2\theta_0(2u_\phi + {u_\phi}^2)$, but it can also be written in terms of more standard oceanographic constants, $u_0 = \dot{\phi}_0 R\cos\theta_0$, $f_0 = 2\omega\sin\theta_0$, and $\beta = 2\omega\cos\theta_0/R$. Using these values,

$$k^2 = 4\left(\frac{u_0\beta}{f_0^2} + \frac{u_0^2}{R^2 f_0^2}\right).$$

The modulus separates inverse elliptic integrals from inverse trigonometric integrals. Notice that if the modulus were 0, equation (28) would be an inverse trigonometric integral, whereas if $k = 1$, equation (28) would be the integral for an inverse hyperbolic function. The modulus depends on both the energy via the u_0^2 term and the angular momentum via the u_0 term. The modulus is dependent on θ_0, the maximum latitude achieved by the particle rather than θ_{mid}, the mid-latitude where $u = 0$, but using the two conservation properties one can express k in terms of θ_{mid} if desired.

A.1. Case $k^2 < 1$

When $k^2 < 1$, the integral in equation (28) is exactly the definition of the inverse elliptic function $\mathrm{dn}^{-1}(u,k)$ (Byrd *et al.* 1971) and therefore

$$\theta_1(t) = \sin^{-1}\left(\sin\theta_0\ \mathrm{dn}(\omega' t, k)\right), \qquad (29)$$

where $\omega' = \sin\theta_0\omega$. The solution to $\phi(t)$ is found using equation (24),

$$\phi(t) = \cos^2\theta_0(\dot{\phi}_0+\omega)\int \frac{1}{1-\sin^2\theta_0\,\mathrm{dn}^2(\omega' t)}\,dt - \omega t. \qquad (30)$$

We can write $\mathrm{dn}^2 u$ in terms of $\mathrm{sn}^2 u$ using the relation $\mathrm{dn}^2 u = 1 - k^2\,\mathrm{sn}^2 u$ to rewrite equation (30),

$$\phi(t) = \frac{1+u_\phi}{\sin\theta_0}\int_0^{\omega' t}\frac{ds}{1+k^2\tan^2\theta_0\,\mathrm{sn}^2 s} - \omega t. \qquad (31)$$

The integral is exactly the definition of an elliptic integral of the third kind (Byrd *et al.* 1971) and thus,

$$\phi_1(t) = \frac{(1+u_\phi)}{\sin\theta_0}\Pi\left(\mathrm{am}\,(\omega' t)\,, -k^2\tan^2\theta_0, k\right) - \omega t \qquad (32)$$

where $\mathrm{am}(u,k)$ is the Jacobi amplitude such that, e.g., $\sin(\mathrm{am}(u,k)) = \mathrm{sn}(u,k)$. The solution $(\theta_1(t),\phi_1(t))$ oscillates between $\cos^{-1}\left((1+u_\phi)\cos\theta_0\right)$ and θ_0 and has a slow westward drift that can be computed by take the integral of equation (30) over one period, similar to using the action-angle variables in Paldor and Sigalov (2001).

The frequency of oscillation occurs at $f = 2\omega\sin\theta$ using the f-plane approximation, but can be computed exactly by noting that the dn function is $2K(k)$ periodic where $K(k)$ is the complete elliptic function and dependent on the modulus. Doing this we find that,

$$f_1(\theta_0) = \frac{\pi\omega\sin\theta_0}{K(k)}. \qquad (33)$$

The u component of velocity follows from equation (24) by noting that

$$R\cos\theta(t)\left(u(t)+\omega R\cos\theta(t)\right) = R^2\cos^2\theta_0\left(\dot{\phi}_0+\omega\right)$$

and then solving for $u(t)$, while v component is found by differentiating $\theta(t)$ with respect to t. The velocity for the first solution is therefore

$$\begin{bmatrix} u_1(t) \\ v_1(t)\end{bmatrix} = \frac{1}{\sqrt{1+k^2\tan^2\theta_0\,\mathrm{sn}^2(\omega' t,k)}} \cdot \begin{bmatrix} u_0 - 2\left(u_0+\frac{u_0^2}{R^2\beta}\right)\mathrm{sn}^2(\omega' t,k) \\ -2\left(u_0+\frac{u_0^2}{R^2\beta}\right)\mathrm{sn}(\omega' t,k)\,\mathrm{cn}(\omega' t,k)\end{bmatrix} \qquad (34)$$

A.2. Case $k^2 = 1$

If we set $k^2 = 1$, then equation (28) reduces to the usual integral for the inverse sech function, which can also be seen from equation (29) because $\mathrm{dn}(u,1) = \mathrm{sech}(u)$. The solution for $\theta(t)$ is therefore,

$$\theta_2(t) = \sin^{-1}\left(\sin\theta_0\ \mathrm{sech}(\omega' t)\right), \qquad (35)$$

and $\phi(t)$ follows from the same integration of equation (24),

$$\phi_2(t) = \omega(1-\cos\theta_0)t + \tan^{-1}\left(\tan\theta_0\tanh(\omega' t)\right). \qquad (36)$$

This is clearly a special case that requires a launch of the particle at precisely the right velocity, $u = \omega R(1-\cos\theta_0)$. In this case the particle hits the equator, essentially remaining in orbit around the equator for its lifetime.

Using the same method as the first case, the velocity of the particle for the second solution is

$$\begin{bmatrix} u_2(t) \\ v_2(t)\end{bmatrix} = \frac{1}{\sqrt{1+k^2\tan^2\theta_0\tanh^2(\omega')}} \cdot \begin{bmatrix} u_0 - \frac{f_0^2}{2\beta}\tanh^2(\omega' t) \\ -\frac{f_0^2}{2\beta}\tanh(\omega' t)\,\mathrm{sech}(\omega' t)\end{bmatrix}. \qquad (37)$$

A.3. Case $k^2 > 1$

The modulus k of a Jacobi elliptic functions is assumed to be $0 < k < 1$, but in this case k exceeds 1. We can work around this issue by using the reciprocal of the modulus, k^{-1}, instead. Because $\mathrm{dn}(u,k) = \mathrm{cn}\left(ku, k^{-1}\right)$, it follows that

$$\theta_3(t) = \sin^{-1}\left(\sin\theta_0 \ \mathrm{cn}\left(k\omega' t, k^{-1}\right)\right), \tag{38}$$

and the $\phi(t)$ solution for this case proceeds in the same way as the $k^2 < 1$ case,

$$\phi_3(t) = \frac{(1+u_\phi)}{k\sin\theta_0}\Pi\left(\mathrm{am}\left(k\omega' t\right), -\tan^2\theta_0, k^{-1}\right) - \omega t. \tag{39}$$

The solution $(\theta_3(t), \phi_3(t))$ oscillates between θ_0 and $-\theta_0$, and has either an eastward or an westward drift depending on the initial conditions.

The cn function is $4K(k)$ periodic and therefore the frequency of oscillation is,

$$f_3(\theta_0) = \frac{k\pi\omega\sin\theta_0}{2K\left(k^{-1}\right)}. \tag{40}$$

Finally, the velocity of the particle is

$$\begin{bmatrix} u_3(t) \\ v_3(t) \end{bmatrix} = \frac{1}{\sqrt{1+\tan^2\theta_0\,\mathrm{sn}^2\left(k\omega' t, k^{-1}\right)}} \cdot \begin{bmatrix} u_0 - \frac{f_0^2}{2\beta}\mathrm{sn}^2\left(k\omega' t, k^{-1}\right) \\ -\frac{f_0^2}{2\beta}k\,\mathrm{sn}\left(k\omega' t, k^{-1}\right)\mathrm{cn}\left(k\omega' t, k^{-1}\right) \end{bmatrix}. \tag{41}$$

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