Stokes Drift of a Rossby Wave

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A plane wave Rossby wave has a sea-surface height

$$\eta(x,y,t) = N_0 \cos(kx + ly - \omega t + \phi) \tag{1}$$

with required dispersion relation

$$\omega = -\frac{\beta_0 k}{k^2 + l^2 + L_R^{-2}} \tag{2}$$

and a velocity field of

$$u = -\frac{g}{f_0}\eta_y, \; v = \frac{g}{f_0}\eta_x. \tag{3}$$

1 Stokes Drift

The stokes drift velocity, U^{S}, is computed by first finding the particle position

$$\xi^x = x_0 - \frac{g}{f_0}\int \eta_y \, dt \tag{4}$$

$$\xi^y = y_0 + \frac{g}{f_0}\int \eta_x \, dt \tag{5}$$

and computing the difference between the Lagrangian and Eulerian velocity.

$$U^{\mathrm{S}} = u\left(\xi^x, \xi^y, t\right) - u(x_0, y_0, t) \tag{6}$$

$$\approx u(x_0,y_0,t) + \left(\xi^x - x_0\right)\frac{\partial u}{\partial x}(x_0,y_0,t) + \left(\xi^y - y_0\right)\frac{\partial u}{\partial y}(x_0,y_0,t) - u(x_0,y_0,t) \tag{7}$$

$$\approx \left[-\frac{g}{f_0}\int \eta_y \, dt\right]\left[-\frac{g}{f_0}\eta_{xy}\right] + \left[\frac{g}{f_0}\int \eta_x \, dt\right]\left[-\frac{g}{f_0}\eta_{yy}\right] \tag{8}$$

$$\approx \frac{g^2}{f_0^2}\left[\eta_{xy}\int \eta_y \, dt - \eta_{yy}\int \eta_x \, dt\right] \tag{9}$$

One still needs to compute the time average over one wave period.

2 Single Plane Wave

We need to compute the following four quantities,

$$\eta_{xy} = - N_0 kl \cos\left(kx + ly - \omega t + \phi\right) \tag{10}$$

$$\eta_{yy} = - N_0 l^2 \cos\left(kx + ly - \omega t + \phi\right) \tag{11}$$

$$\int \eta_y \, dt = - N_0 \frac{l}{\omega} \cos\left(kx + ly - \omega t + \phi\right) \tag{12}$$

$$\int \eta_x \, dt = - N_0 \frac{k}{\omega} \cos\left(kx + ly - \omega t + \phi\right). \tag{13}$$

If you use these values in equation 9, you find that to lowest order, $U^{\mathrm{S}} = 0$. A single Rossby wave does not induce stokes drift.

3 Two Plane Waves

The next level of complication is to add a second plane wave and see if we can get Stokes drift. Now take something of the form,

$$\eta(x, y, t) = N_0 \cos\left(k_0 x + l_0 y - \omega_0 t\right) + N_1 \cos\left(k_1 x + l_1 y - \omega_1 t + \phi\right) \tag{14}$$

From this we can compute that,

$$\eta_{xy} = - N_0 k_0 l_0 \cos\theta_0 - N_1 k_1 l_1 \cos\theta_1 \tag{15}$$

$$\eta_{yy} = - N_0 l_0^2 \cos\theta_0 - N_1 l_1^2 \cos\theta_1 \tag{16}$$

$$\int \eta_y \, dt = - N_0 \frac{l_0}{\omega_0} \cos\theta_0 - N_1 \frac{l_1}{\omega_1} \cos\theta_1 \tag{17}$$

$$\int \eta_x \, dt = - N_0 \frac{k_0}{\omega_0} \cos\theta_0 - N_1 \frac{k_1}{\omega_1} \cos\theta_1 \tag{18}$$

The first term piece gives us,

$$\eta_{xy} \int \eta_y \, dt = \left[-N_0 k_0 l_0 \cos\theta_0 - N_1 k_1 l_1 \cos\theta_1\right] \left[-N_0 \frac{l_0}{\omega_0} \cos\theta_0 - N_1 \frac{l_1}{\omega_1} \cos\theta_1\right] \tag{19}$$

$$= N_0^2 \frac{k_0 l_0^2}{\omega_0} \cos^2\theta_0 + N_1^2 \frac{k_1 l_1^2}{\omega_1} \cos^2\theta_1 + N_0 N_1 \left(\frac{k_0 l_0 l_1}{\omega_1} + \frac{k_1 l_0 l_1}{\omega_0}\right) \cos\theta_0 \cos\theta_1 \tag{20}$$

while the second term piece gives us,

$$\eta_{yy} \int \eta_x \, dt = \left[-N_0 l_0^2 \cos\theta_0 - N_1 l_1^2 \cos\theta_1 \right] \left[-N_0 \frac{k_0}{\omega_0} \cos\theta_0 - N_1 \frac{k_1}{\omega_1} \cos\theta_1 \right] \tag{21}$$

$$= N_0^2 \frac{k_0 l_0^2}{\omega_0} \cos^2\theta_0 + N_1^2 \frac{k_1 l_1^2}{\omega_1} \cos^2\theta_1 + N_0 N_1 \left(\frac{k_1 l_0^2}{\omega_1} + \frac{k_0 l_1^2}{\omega_0} \right) \cos\theta_0 \cos\theta_1 \tag{22}$$

The isolated plane wave terms cancel, as they did before, but the cross terms survive. So we have that,

$$U^{\mathrm{S}} \approx \frac{g^2}{f_0^2} \left[\eta_{xy} \int \eta_y \, dt - \eta_{yy} \int \eta_x \, dt \right] \tag{23}$$

$$\approx \frac{g^2 N_0 N_1}{f_0^2} \left(\frac{k_0 l_0 l_1}{\omega_1} + \frac{k_1 l_0 l_1}{\omega_0} - \frac{k_1 l_0^2}{\omega_1} - \frac{k_0 l_1^2}{\omega_0} \right) \cos\theta_0 \cos\theta_1 \tag{24}$$

This computation still depends on the time, t, buried in θ_0 and θ_1. When we time average over one wave period (or the least common multiple of the two wave periods), the average will sum to zero unless the wave periods are the same. If we explicitly include the phase ϕ that we buried in θ_1, then the integral in question is,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \cos\theta_0 \cos(\theta_1 + \phi) \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos\theta_0 \left[\cos\theta_1 \cos\phi - \sin\theta_1 \sin\phi \right] \, dt$$

$$= \frac{\cos\phi}{\pi} \delta_{\omega_1 \omega_2} \tag{25}$$

Thus, we can say that $\omega_1 = \omega_2$ and just call it it ω. Equation 24 can then be rewritten as,

$$U^{\mathrm{S}} \approx \frac{g^2 N_0 N_1}{2 f_0^2} \cos\phi \frac{1}{\omega} (l_0 - l_1)(k_0 l_1 - k_1 l_0). \tag{26}$$

Equation 26 has a nice geometrical interpretation. Recall the dispersion diagram from section 3.22 in Pedlosky which shows and isoline of ω in k, l space. In all cases that isoline is a circle. Because both of our plane waves have the same ω, their wave vectors must point to that isocircle. The $k_0 l_1 - k_1 l_0$ part of equation 26 is a jacobian – a measure of the area of the parallelogram of the two wave vectors. This means that those two wave vectors cannot be parallel if we are to have any stokes drift. Additionally, the factor of factor of $l_0 - l_1$ means that we have to have two different meridional wave numbers.

Consider the case where the wavevectors point to opposite sides of the isocircle centered at k_c. That is to say that,

$$(k_0, l_0) = (k_c + r\cos\alpha, r\sin\alpha) \tag{27}$$

$$(k_1, l_1) = (k_c - r\cos\alpha, -r\sin\alpha) \tag{28}$$

where $r = \sqrt{k_c^2 - L_R^2}$ is the radius of the isocircle and α is the angle. With this choice of wave vectors, equation 26 becomes,

$$U^{\mathrm{S}} \approx 4\frac{g^2 N_0 N_1}{f_0^2} \cos\phi \frac{k_c^2}{\beta_0} \left(k_c^2 - L_R^{-2}\right) \sin^2\alpha. \tag{29}$$

This has an extremum at $k_c = \sqrt{\frac{1}{2L_R}}$ which gives us,

$$U^{\mathrm{S}} \approx -\frac{g^2 N_0 N_1}{f_0^2 L_R^2} \frac{\cos\phi}{\beta_0 L_R^2} \sin^2\alpha. \tag{30}$$

This is maximized when $\alpha = \frac{\pi}{2}$, in other words, when the two wave vectors point to the top and bottom of the isocircle. This means the stokes drift is always westward.

To get a sense of the size of this, let's take some values for latitude 24. At latitude 24, $f_0 = 5.9 \cdot 10^{-5}\,\mathrm{s}^{-1}$, $\beta = 2.1 \cdot 10^{-11}\,\mathrm{m}^{-1}\mathrm{s}^{-1}$, $c_p(0,0) = 4.7\,\mathrm{cm\,s}^{-1}$, $L_R = 4.72 \cdot 10^4$ km. Say $N_0 = N_1 = 1$ cm and $\phi = 0$. One can write the above as a ratio of two speeds,

$$u_g = \frac{g N_0}{f_0 L_R} = 3.5\mathrm{cm/s} \tag{31}$$

$$c_p = \beta_0 L_R^2 = 4.7\mathrm{cm/s} \tag{32}$$

where the stokes drift is then

$$U^{\mathrm{S}} \approx -\frac{u_g^2}{c_p} \approx -2.7 \text{ cm/s}. \tag{33}$$

3.1 Meridional Drift

Now consider the meridional Stokes drift. Equation 9 becomes,

$$V^{\mathrm{S}} \approx \frac{g^2}{f_0^2} \left[\eta_{xy} \int \eta_x \, dt - \eta_{xx} \int \eta_y \, dt\right] \tag{34}$$

and equation 26 becomes

$$V^{\mathrm{S}} \approx -\frac{g^2 N_0 N_1}{2 f_0^2} \frac{1}{\omega} (k_0 - k_1)(k_0 l_1 - k_1 l_0). \tag{35}$$

The interpretation is similar to the zonal drift, but we need to maximize the area between the wavenumber vectors but also keep a large separation between k_0 and k_1. With the same choice of wave vectors as last time, the meridional stokes drift becomes,

$$V^{\mathrm{S}} \approx 4\frac{g^2 N_0 N_1}{f_0^2} \frac{k^2}{\beta_0} \left(k^2 - L_R^{-2}\right) \sin\alpha \cos\alpha. \tag{36}$$

This has an extremum at $k = \sqrt{\frac{1}{2L_R}}$ which gives us,

$$V^{\mathrm{S}} \approx -\frac{g^2 N_0 N_1}{f_0^2 L_R^2} \frac{1}{\beta_0 L_R^2} \sin\alpha\cos\alpha. \tag{37}$$

This time the meridional drift has two extrema, one at $\alpha = \frac{\pi}{4}$ and the other at $\alpha = -\frac{\pi}{4}$. The meridional stokes drift is thus,

$$V^{\mathrm{S}} \approx \pm\frac{1}{2}\frac{u_g^2}{c_p} \approx \pm 1.3 \text{ cm/s}. \tag{38}$$